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Abstract. New uncertainty relations for n observables are established. The relations take the invariant form of inequalities between the characteristic coefficients of order r, $r = 1, 2, \ldots, n$, of the uncertainty matrix and the matrix of mean commutators of the observables. It is shown that the second- and third-order characteristic inequalities for the three generators of $SU(1,1)$ and $SU(2)$ are minimized in the corresponding group-related coherent states with maximal symmetry.

1. Introduction

The uncertainty relations (UR) are basic nonclassical features of quantum theory. In recent decades they have been extensively used in quantum optics for constructing the so-called nonclassical states [1]. In 1927 Heisenberg [2] formulated the uncertainty principle as the impossibility to determine simultaneously the position q and momentum p of a particle with an accuracy higher than the Plank constant $\hbar$: the product of the uncertainties Δp, Δq in p and q should not be less than $\hbar$, $\Delta p \Delta q \sim \hbar$. It was Weyl [2] who proved the 'Heisenberg uncertainty relation' for p and q ($\Delta^2 p \Delta^2 q \geqslant \hbar^2/4$) and Robertson [3] who extended the latter to arbitrary two quantum observables (Hermitian operators) X and Y,

$$\Delta^2 X \Delta^2 Y \geqslant \tfrac{1}{4}|\langle [X, Y]\rangle|^2 \tag{1}$$

where $[X, Y]$ is the commutator of X and Y, $\langle X\rangle$ is the average value of X and $\Delta^2 X$ is the variance of X. Nevertheless, the inequality (1) is referred to as Heisenberg UR for two observables. A more precise inequality for two observables was established by Schrödinger [4],

$$\Delta^2 X \Delta^2 Y - (\Delta XY)^2 \geqslant \tfrac{1}{4}|\langle [X, Y]\rangle|^2 \tag{2}$$

where $\Delta XY = \langle XY + YX\rangle/2 - \langle X\rangle\langle Y\rangle$ is the covariance of X and Y. The coherent states (CS) [5] and squeezed states [1] of the one-mode radiation field widely discussed in the literature are pure quantum states in which the inequalities (1) and/or (2) for the two canonical observables p and q ($[p, q] = -\mathrm{i}\hbar$) are minimized [5–7]. The minimization of the Schrödinger UR for two observables was considered in [8], where the minimizing states for two generators of the groups $SU(1,1)$ and $SU(2)$ have also been constructed and discussed.

An important advantage of the Schrödinger formulation of the uncertainty principle is the invariance of the equality in (2) under linear nondegenerate transformations of X and Y [9, 10], in particular under linear canonical transformations of p and q [6, 11, 10]. Robertson [12] extended UR (1) to arbitrary n observables X_j, $j = 1, 2, \ldots,$ in the invariant form

$$\det \sigma(\boldsymbol{X}; \rho) \geqslant \det C(\boldsymbol{X}; \rho) \tag{3}$$

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where $\sigma(\boldsymbol{X};\rho)$ is the uncertainty (the dispersion or the covariance) matrix of n observables in the (generally mixed) state ρ, $\sigma_{jk} = \langle X_jX_k + X_kX_j\rangle/2 - \langle X_j\rangle\langle X_k\rangle$, and $C(\boldsymbol{X};\rho)$ is the $n\times n$ matrix of mean values of the commutators $[X_j, X_k]$ times a factor $1/2\mathrm{i}$, $C_{jk} = (-\mathrm{i}/2)\langle[X_j, X_k]\rangle$. The uncertainty matrix $\sigma(\boldsymbol{X})$ is symmetric, and the mean commutator matrix $C(\boldsymbol{X})$ is antisymmetric. The other important advantage of Robertson UR is in its free integer parameter n. This enables one to treat more complicated algebras (of observables) with any finite dimension.

The aim of this paper is to establish a new series of uncertainty relations, which we call the *characteristic UR*. The Robertson UR appears as one of the family of characteristic UR.

The idea is to consider the matrices σ and C as matrices of linear maps in the n-dimensional vector space E_n, spanned by the operators X_j, $j = 1, 2, \ldots, n$. The quantities $\det\sigma$ and $\det C$, which enter the Robertson UR, appear as characteristic coefficients $C_n^{(n)}(\sigma)$ and $C_n^{(n)}(C)$ of the characteristic polynomials of σ and C and the natural question arises whether there are inequalities which relate the other characteristic coefficients $C_r^{(n)}(\sigma)$ and $C_r^{(n)}(C)$, $r = 0, 1, \ldots, n$. The answer turned out to be positive and the corresponding characteristic inequalities are established below.

2. Characteristic uncertainty relations

In order to extend the Robertson UR to the other characteristic coefficients let us first recall [10] the transformation properties of the dispersion matrix $\sigma(\boldsymbol{X})$ and the matrix of mean commutators $C(\boldsymbol{X})$ under linear transformations of the operators X_j,

$$X_j \to X_j' = \lambda_{jk}X_k \qquad \text{or in matrix form } \boldsymbol{X}' = \Lambda\boldsymbol{X}. \tag{4}$$

Using the definition of σ and C we easily obtain

$$\sigma(\boldsymbol{X}') \equiv \sigma' = \Lambda\sigma\Lambda^{\mathrm{T}} \qquad C' \equiv C(\boldsymbol{X}') = \Lambda C\Lambda^{\mathrm{T}} \tag{5}$$

where Λ^{T} is the transposed of Λ. If the transformation matrix is real and nonsingular then the new n operators X_j' are again Hermitian and σ' is their dispersion matrix.

The transformation law (5) ensures the invariance of the equality in (3) under nonsingular linear transformations of observables (4), $\Lambda \in GL(n, R)$. If the operators X_j close a Lie algebra L, then the equality in the Robertson UR is invariant under the transformations of the group $\mathrm{Aut}(L)$ of automorphisms of L. It is curious that the equality in (3) is invariant under a wide class of nonlinear state-dependent transformations of X_j. Such are the transformations (5) with $\Lambda = \sigma$ or $\Lambda = C$ in cases of $\det C > 0$.

With the aim of establishing new uncertainty inequalities let us now consider the characteristic equations for matrices σ and C (λ and μ are parameters),

$$0 = \det(\sigma - \lambda) = \sum_{r=0}^{n} C_r^{(n)}(\sigma)(-\lambda)^{n-r} \qquad 0 = \det(C - \mu) = \sum_{r=0}^{n} C_r^{(n)}(C)(-\mu)^{n-r}. \tag{6}$$

These equations are invariant under similarity transformations $\sigma \to \Lambda\sigma\Lambda^{-1}$, $C \to \Lambda C\Lambda^{-1}$. The invariant coefficients $C_r^{(n)}(\sigma)$ ($C_r^{(n)}(C)$), $r = 0, 1, \ldots, n$, in (6) are called the *characteristic coefficients* of σ (C) [13]. If one treats σ and C as linear maps in an n-dimensional vector space E ($E \ni \boldsymbol{y} = \sigma\boldsymbol{x}$, $\boldsymbol{x} \in E$), then $\sigma' = \Lambda\sigma\Lambda^{-1}$ and $C' = \Lambda C\Lambda^{-1}$ represent the same maps in the new basis of E (related to the old one by means of matrix $(\Lambda^{\mathrm{T}})^{-1}$). The characteristic coefficients $C_r^{(n)}(\phi)$ of a matrix ϕ are equal [13] to the sum of all principle minors $M(i_1,\ldots,i_r;\phi)$ of order r,

$$C_r^{(n)}(\phi) = \sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant n} \begin{vmatrix} \phi_{i_1i_1}\phi_{i_1i_2}\ldots\phi_{i_1i_r} \\ \phi_{i_2i_1}\phi_{i_2i_2}\ldots\phi_{i_2i_r} \\ \ldots\ldots\ldots\ldots \\ \phi_{i_ri_1}\phi_{i_ri_2}\ldots\phi_{i_ri_r} \end{vmatrix} \equiv \sum_{1\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant n} M(i_1,\ldots,i_r;\phi). \tag{7}$$

One has $C_0^{(n)} = 1$, $C_1^{(n)} = \mathrm{Tr}\,\phi = \sum\phi_{ii}$ and $C_n^{(n)} = \det\phi$. For $n = 3$ we have, for example, three principle minors of order 2, i.e.

$$C_2^{(3)}(\phi) = \begin{vmatrix} \phi_{11}\phi_{12} \\ \phi_{21}\phi_{22} \end{vmatrix} + \begin{vmatrix} \phi_{11}\phi_{13} \\ \phi_{31}\phi_{33} \end{vmatrix} + \begin{vmatrix} \phi_{22}\phi_{23} \\ \phi_{32}\phi_{33} \end{vmatrix} \tag{8}$$

where $|\phi| \equiv \det\phi$ for a matrix ϕ. In these notations Robertson UR (3) reads $C_n^{(n)}(\sigma(\boldsymbol{X})) \geqslant C_n^{(n)}(C(\boldsymbol{X}))$ for any quantum state. We shall now show that similar inequalities hold for the other characteristic coefficients of σ and C, namely

$$C_r^{(n)}(\sigma(\boldsymbol{X})) \geqslant C_r^{(n)}(C(\boldsymbol{X})) \qquad r = 1, 2, \ldots, n. \tag{9}$$

The key observation to this aim is that the principle submatrices $\sigma(X_{i_1},\ldots,X_{i_r};\rho)$, $i_1 < i_2 < \cdots < i_r$,

$$\sigma(X_{i_1},\ldots,X_{i_r};\rho) = \begin{pmatrix} \sigma_{i_1i_1}\sigma_{i_1i_2}\ldots\sigma_{i_1,i_r} \\ \sigma_{i_2i_1}\sigma_{i_2i_2}\ldots\sigma_{i_2i_r} \\ \ldots\ldots\ldots \\ \sigma_{i_ri_1}\sigma_{i_ri_2}\ldots\sigma_{i_ri_r} \end{pmatrix} \qquad M(i_1,\ldots,i_r;\sigma) = \det\sigma(X_{i_1},\ldots,X_{i_r}) \tag{10}$$

can be regarded as uncertainty matrix for r observables $X_{i_1},\ldots,X_{i_r}$ with $C(X_{i_1},\ldots,X_{i_r};\rho)$ as the corresponding mean commutator matrix. Therefore the inequality (3) holds for the principle minors as well:

$$\det\sigma(X_{i_1},\ldots,X_{i_r};\rho) \geqslant \det C(X_{i_1},\ldots,X_{i_r};\rho). \tag{11}$$

It is worth recalling now that $\sigma(\boldsymbol{X})$ and $C(\boldsymbol{X})$ are non-negative definite [10] and therefore all of their principle minors are also non-negative [13],

$$M(i_1,\ldots,i_r;\sigma) \geqslant 0 \qquad M(i_1,\ldots,i_r;C) \geqslant 0. \tag{12}$$

From (12) and (7) we derive that the inequalities (9) do hold. We shall call these inequalities the *characteristic UR* for n observables (Hermitian operators). The two sides of these relations are invariant under similarity transformations $\sigma \to \Lambda\sigma\Lambda^{-1}$ and $C \to \Lambda C\Lambda^{-1}$. The transformed matrix $\sigma' = \Lambda\sigma\Lambda^{-1}$ can be considered as an uncertainty matrix for new n observables X'_j in the same state ρ iff $\Lambda^{-1} = \Lambda^{\mathrm{T}}$ as is seen from (5). Therefore the characteristic UR (9) are invariant under linear transformation of the observables with orthogonal Λ.

On the other hand, there are trace class invariant coefficients, related to any $n \times n$ matrix, and one can look for uncertainty inequalities involving these coefficients for the physical matrices σ and C. A series of such inequalities for positive definite $2N \times 2N$ dispersion matrices for $2N$ observables are established in [10] (the particular case of canonical observables being considered in [11]),

$$\mathrm{Tr}(\mathrm{i}\sigma(\boldsymbol{X},\rho)J)^{2k} \geqslant 2^{1-2k}\sum_{j=1}^{N}|\langle[X'_\nu, X'_{N+\nu}]\rangle|^{2k} \qquad k = 1, 2, \ldots, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{13}$$

where $X'_j = \Lambda(\rho)_{jl}X_l$, Λ being the symplectic matrix which diagonalizes $\sigma(\boldsymbol{X};\rho)$ [10]. The traces $\mathrm{Tr}(\mathrm{i}\sigma(\boldsymbol{X},\rho)J)^{2k}$ are invariant under symplectic transformations Λ. If the operators X_j close a Lie algebra L with a Cartan–Killing tensor g then $\mathrm{Tr}(\sigma(\boldsymbol{X},\rho)g)^{2k}$ is invariant under the group of automorphisms of L. At $k = 1 = N$ one has $\mathrm{Tr}(\mathrm{i}\sigma(\boldsymbol{X},\rho)J)^{2k} = \det\sigma(\boldsymbol{X},\rho)$ and (13) coincides with (2).

3. Minimization of the characteristic uncertainty relations

The minimization of inequalities (1) and (2) proved useful in constructing states with interesting physical and mathematical properties. States which minimize a certain uncertainty relation will be called minimum uncertainty states (MUS) or intelligent states (see [8, 10] and references therein). States which minimize (2) (or (1)) are called Schrödinger (Heisenberg) MUS or intelligent states (or correlated coherent states [6]). For any pair of observables X, Y the necessary and sufficient condition for a state $|\Psi\rangle$ to minimize Schrödinger UR (2) is $|\Psi\rangle$ to be an eigenstate of a complex combination of X, Y [8],

$$(uA + vA^\dagger)|\Psi\rangle = z|\Psi\rangle \qquad A = X + \mathrm{i}Y \qquad u, v \in C. \tag{14}$$

For X, Y being the quadratures p, q of boson annihilation operator a Schrödinger MUS coincide [7] with standard (or canonical) squeezed states [1]. The family of Schrödinger MUS $|z, u, v; k\rangle$ for the two quadratures of the Weyl lowering operator K_- for the $su(1,1)$ algebra was constructed in [8] using the analytic representation of Barut and Girardello [14].

The minimization of Robertson UR was studied in [10]. Robertson MUS exist for a broad class of physical systems. It was shown [10] that group-related CS with maximal symmetry for semisimple Lie groups are Robertson MUS for the quadratures of Weyl lowering operators. For an odd number n of observables X_i a necessary and sufficient condition for a state $|\Psi\rangle$ to minimize (3) is $|\Psi\rangle$ to be an eigenstate of a real linear combination of all observables. This condition remains sufficient for even n, $n = 2N$, as well. For even n another sufficient condition is $|\Psi\rangle$ to be an eigenstate of N *complex* combinations of X_i,

$$(\beta_{\alpha i}X_i)|\Psi\rangle \equiv (u_{\alpha\beta}A_\beta + v_{\alpha\beta}A^\dagger_\beta)|\Psi\rangle = z_\alpha|\Psi\rangle \tag{15}$$

where $i = 1, \ldots, n$, $\alpha, \beta = 1, 2, \ldots, N$, $A_\alpha = X_\alpha + \mathrm{i}X_{\alpha+N}$ and summation over repeated indices is adopted. The above conditions are satisfied by the N-mode canonical CS ($u = 1$, $v = 0$, $A_\alpha = a_\alpha$); by canonical squeezed states ($uu^\dagger - vv^\dagger = 1$, $A_\alpha = a_\alpha$) and by canonical even/odd CS ($u = 1$, $v = 0$, $A_\alpha = a_\alpha^2$), i.e. these important in quantum optics states are Robertson MUS for the quadratures of all a_α and a_α^2 correspondingly [10].

Using the structure (7) of the characteristic coefficients and the non-negativity of the principle minors involved one can easily establish the following minimization conditions for (9).

Proposition 1. The rth-order characteristic UR (9) is minimized in a state $|\Psi\rangle$ if $|\Psi\rangle$ is a Robertson MUS for every set of r observables $X_{i_1}, X_{i_2}, \ldots, X_{i_r}$, $1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n$.

The characteristic UR of maximal order $r = n$ is that of Robertson and its minimization conditions were listed just above [10].

Now it is of principle importance to show that the rth-order characteristic MUS for $r \leqslant n$ do exist. It is clear that the first-order characteristic UR is trivial since it reads $\sigma_{11} + \sigma_{22} + \cdots + \sigma_{nn} \geqslant 0$, where all variances $\sigma_{ii} \equiv \Delta^2 X_i$ are non-negative in any state. Its minimization is also clear—it is minimized in the common eigenstates of X_i only. The case of $r = n$ was examined in [10]. So we have to provide the minimization of rth-order characteristic UR, $2 \leqslant r < n$.

To this aim let us consider the family of $su(1,1)$ algebra related CS $|z,u,v,w;k\rangle$, which are eigenstates of the general element of the algebra in the representations $D^+(k)$, $k=\frac{1}{4},\frac{3}{4}$ and $k=\frac{1}{2},1,\ldots,$

$$(uK_- + vK_+ + wK_3)|z,u,v,w;k\rangle = z|z,u,v,w;k\rangle. \tag{16}$$

Here $K_\pm = K_1 \pm \mathrm{i}K_2$ are Weyl raising and lowering operators and $K_{1,2,3}$ are Hermitian generators of the group $SU(1,1)$. The solution to equation (16) was obtained in [9, 15, 10], the case $w=0$ being solved previously in [8]. The large family of $|z,u,v,w;k\rangle$ contains all $SU(1,1)$ group-related CS with symmetry (see [5] and references therein), the Schrödinger K_1–K_2 MUS $|z,u,v,w=0;k\rangle \equiv |z,u,v;k\rangle$ containing the group-related CS with maximal symmetry [8]. In the limit $u=1$, $v=0=w$ the CS $|z;k\rangle$ of Barut and Girardello [14] are reproduced: $|z;k\rangle = |z,u=1,v=0,w=0;k\rangle$. In the analytic Barut–Girardello (BG) representation (which can be shown to be valid for $k=\frac{1}{4},\frac{3}{4}$ as well) the operators K_i act as differential operators,

$$K_+ = \eta \qquad K_- = 2k\frac{\mathrm{d}}{\mathrm{d}\eta} + \eta\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \qquad K_3 = k + \eta\frac{\mathrm{d}}{\mathrm{d}\eta} \tag{17}$$

and the states $|z,u,v,w:k\rangle$ are represented (for $u \neq 0$) by the analytic functions

$$\Phi_z(\eta;u,v,w) = N\mathrm{e}^{c\eta}\,{}_1F_1(a,b;c_1\eta) \tag{18}$$

where N is the normalization factor, $a = k + z/l$, $b=2k$, $c = -(w+l)/2u$, $c_1 = l/u$, $l \equiv \sqrt{w^2-4uv}$ and ${}_1F_1(a,b,z)$ is the confluent hypergeometric function (the Kummer function) [16]. The limit $l=0$ can be easily taken in (18), and the more simple case of $u=0$ should be treated separately [9, 15, 10].

For the three observables $K_{1,2,3}$ we have two nontrivial characteristic UR, namely those for $r=2$ and $r=n=3$ in (9). The Robertson relation for the three operators $K_{1,2,3}$ is minimized in $|z,u,v,w;k\rangle$ for $v=u^*$ and real w and z [10]. Such normalized intelligent states are, for example, the $SU(1,1)$ group-related CS $|\zeta;k\rangle$ with maximal symmetry,

$$|\zeta;k\rangle = N\exp(\zeta K_+)|k,k\rangle \qquad \zeta\in C,\ |\zeta|<1.$$

These CS, which in the BG representation are represented by the analytic function $\exp(\zeta\eta)$ of the variable η [8], obey equation (16) with $v^*=u$, $\arg u = -\arg\zeta$, $w=w^*=|u|(1/|\zeta|-|\zeta|)$ and $z=z^*=k|u|(-|\zeta|+1/|\zeta|)$, with u remaining arbitrary. We are now ready to prove that $|\zeta;k\rangle$ minimize the second-order UR as well.

According to proposition 1, a state $|\Psi\rangle$ minimizes the second-order characteristic UR for the three observables $K_{1,2,3}$ if it is an eigenstate of the three combinations $\beta_1K_1+\beta_2K_2 = uK_-vK_+ + 0K_3$, $\beta_1'K_1+\beta_3'K_3 = u'K_-v'K_+ + w'K_3$ and $\beta_2''K_2+\beta_3'' = u''K_-v''K_+ + w''K_3$, for some real or complex $\beta_{1,2}$, $\beta'_{1,2,3}$ and $\beta''_{1,2,3}$,

$$\begin{aligned}
&(\beta_1K_1+\beta_2K_2)|\Psi\rangle = z|\Psi\rangle\\
&(\beta_1'K_1+\beta_3'K_3)|\Psi\rangle = z'|\Psi\rangle\\
&(\beta_2''K_2+\beta_3''K_3)|\Psi\rangle = z''|\Psi\rangle.
\end{aligned} \tag{19}$$

In the representation (17) the system (19) takes the form of second-order differential equations. One can check that the functions

$$f(\eta,m) = \eta^m\mathrm{e}^{-\zeta\eta} \tag{20}$$

satisfy the system (19) for $m=0$ and for $m=1-2k$ if

$$\beta_1 = \mathrm{i}\beta_2\frac{1-\zeta^2}{1+\zeta^2} \qquad \beta_1' = 2\beta_3'\frac{\zeta}{1+\zeta^2} \qquad \beta_2'' = 2\mathrm{i}\beta_3''\frac{\zeta}{1-\zeta^2} \tag{21}$$

the eigenvalues being $z = (k+m\zeta)2\mathrm{i}\beta_2\zeta^2/(1+\zeta^2)$, $z' = (k+m)\beta_3'(1-\zeta^2)/(1+\zeta^2)$, $z'' = (k+m)\beta_3''(1+\zeta^2)/(1-\zeta^2)$ (with β_3'', β_3' and β_2 remaining arbitrary). This proves that the group-related CS $|\zeta;k\rangle$ are $C_2^{(3)}$ and $C_3^{(3)}$ characteristic MUS. Let us recall that the group $SU(1,1)$ has important (in quantum optics and other fields of quantum theory) one-mode ($k=\frac{1}{4},\frac{3}{4}$) and two-mode ($k=\frac{1}{2},1,\ldots$) boson representations. In the one-mode case the CS $|\zeta;k=\frac{1}{4}\rangle$ coincides with the canonical squeezed vacuum [1].

In a similar manner, using the results of papers [8, 10, 15] for example, one can establish that the $SU(2)$ group-related CS with maximal symmetry (the Bloch CS) are $C_2^{(3)}$ and $C_3^{(3)}$ characteristic MUS.

Thus, the characteristic UR can be used for finer classification of quantum states. For a given algebra they all are within the large set of eigenstates of general algebra element (algebraic CS [9, 15]).

4. Concluding remarks

On the abstract matrix level Robertson proved [12, 17] that if $H = S + \mathrm{i}K$ is non-negative definite Hermitian matrix (where S and K are real) then $\det S \geqslant \det K$. Matrix $\sigma(\boldsymbol{X}) + \mathrm{i}C(\boldsymbol{X})$ is Hermitian and non-negative, therefore $\det\sigma \geqslant \det C$. Using Robertson' result we have proved in the above that if the combination $S+\mathrm{i}K$ of two real matrices S and K is non-negative and Hermitian then the characteristic coefficients of S and K obey the inequalities

$$C_r^{(n)}(S) \geqslant C_r^{(n)}(K) \qquad r = 1, 2, \ldots, n. \tag{22}$$

The importance of the characteristic coefficients of a matrix is surely beyond doubt. In differential geometry and differential topology of fibre bundles with connections they are widely used as generators of topological invariants of the bundles by means of the De Rham cohomology of the corresponding base spaces [18]. In gauge field theories they are also well known and appropriately used [19]. It is our belief that the characteristic inequalities (22) will also be useful in the above-described fields.

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